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Strong-coupling expansion for classical Yang–Mills theory

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Abstract. The classical Yang–Mills theory in the strong-coupling limit is investigated. To do this a formal $1/g$ expansion for potentials is applied. A recurrence formula for the nth-order term in the expansion is given. Mathematical correctness is also discussed and supported by simple examples. The recurrence equations are shown to be gauge covariant order-by-order, and a local screening effect in the strong-coupling limit is established.

1. Introduction

During the last few years the classical Yang–Mills (YM) theory has been investigated intensively. The hope is that analysis of the classical theory will provide insight which will be of use for the attack on the full quantum problem, especially for strong couplings (Dashen *et al* 1974, Rajaraman 1975, Polyakov 1975, Jackiw 1977). This seems to be supported by our understanding of quark confinement where topological, geometrical objects play a crucial role (Polyakov 1977, 't Hooft 1981).

Perturbative quantum chromodynamics (QCD) is successful in describing the short-distance interaction between quarks (high-energy region; see, e.g., Mueller 1981). Little is known, however, about the behaviour of the theory for large quark separations (low-energy, strong-coupling region; Baker 1981).

The behaviour of the classical gauge theory for weak couplings is also known and it has been established that at some critical value of the coupling constant $g = g_{\text{crit}} \approx 1$ there is a 'phase transition' and some new, unknown field configurations become essential for strong couplings. In this (low-energy) region only a few qualitative and numerical results have been obtained (Mandula 1977, Jackiw *et al* 1979, Freedman *et al* 1980, Malec 1980).

For strong couplings the semiclassical approximation seems to be particularly adequate, as was suggested many years ago (Tomonaga 1946). For elementary particle physics this point of view is supported by successful phenomenological string and bag models (Nielsen and Olesen 1973, Chodos *et al* 1974).

In this paper a new approach to the strong-coupling (low-energy) region of the classical YM theory is proposed. Expansion in powers of $1/g$ is used for solving, at least approximately, the YM equations. Our method is particularly suitable for those potentials which are singular at $g = 0$ but analytic at $g = \infty$, where usual perturbation theory is useless.

Only the simplest of such potentials are known explicitly. However, they are most interesting because of their physical relevance.

Similar methods are widely used in hydrodynamics for the Navier–Stokes equation (see, e.g., Fraenkel 1963).

In § 2 a formal $1/g$ expansion for YM potentials is introduced and recurrence equations for arbitrary-order coefficients are obtained. Section 3 is devoted to a brief outline of the mathematical background. In § 4 simple examples are given to motivate the applicability of the formal expansion to the YM equations. Section 5 and the appendix are devoted to proving that the proposed method is gauge covariant order-by-order in $1/g$. In § 6 the local screening effect is established in the large-coupling limit within our approach. Conclusions and perspectives are discussed in the final section.

2. Formal $1/g$ expansion

We assume that an external current $j^a_\mu(x, g)$ has a $1/g$ expansion of the form

$$j^a_\mu(x, g) = \sum_{n=0}^{\infty} \overset{n}{j}{}^a_\mu(x)\varepsilon^n \tag{1'}$$

where $\varepsilon \equiv 1/g$, μ, ν are space–time indices and a, b and c are gauge group ('colour') indices. The italic letters n, i, j, k are reserved for labelling the order of coefficients in the ε expansion.

Furthermore, we make the following assumption about the form of the potential:

$$A^a_\mu(x, g) = \sum_{n=0}^{\infty} \overset{n}{A}{}^a_\mu(x)\varepsilon^n. \tag{1}$$

Then in the strong-coupling limit ($g \to \infty$, $\varepsilon \to 0$) we have

$$A^a_\mu(x, \varepsilon) \to \overset{0}{A}{}^a_\mu(x) + \mathrm{O}(\varepsilon). \tag{2}$$

As is evident from (1), our formal solution is a power series in ε. From the mathematical point of view it is a rather stringent assumption. However, Abelian solutions which have a $1/\varepsilon$ pole do not play any essential role for the strong couplings (Mandula 1977) and in the strong-coupling limit they become infinite. On the other hand, most of the known solutions such as instantons, Wu–Yang monopoles and the magnetic dipole solution (Sikivie and Weiss 1978) are linear functions of ε †. In these cases the expansion (1) is trivially satisfied (in fact, it is reduced to the first one or two terms only).

There are also solutions to the YM equations which have a finite strong-coupling limit, but the method presented is inappropriate for them (e.g., 't Hooft–Polyakov monopoles, dyons, vortices). They are functions of ε of the type $\exp(-1/\varepsilon)$ and they are non-analytic at $\varepsilon = 0$ ($g = \infty$). This seems to be a feature of all YM potentials accompanied by Higgs fields.

Nevertheless it is possible that in the YM theory there are solutions more complicated than those linear in ε and also analytic at $\varepsilon = 0$; an example is $\exp(-\varepsilon)$—this type of coupling dependence is familiar from the theory of superconductivity (Schriffer 1964), which is considered to be closely related to elementary particle physics (Nambu

† This feature is common to sourceless potentials and is due to the simple scaling property of the YM equations: after the transformation $A \to A/g$, the coupling constant g can be rescaled from the homogeneous YM equations. This trick, however, breaks down if there is a non-zero external current and a coupling constant dependence of potentials can be arbitrarily complicated, depending on the form of the external current.

and Jona-Lasinio 1961, 't Hooft 1981). Hence the expansion (1) for YM potentials seems to be not only mathematically plausible but also physically relevant.

We limit ourselves to the static ($\partial_0 A^a = 0$) or Euclidean YM potentials. For the sake of simplicity we shall consider the gauge group G = SU(2). The field equations have the form

$$\varepsilon^2(\Delta A^a_\mu - \partial_\mu \partial^\nu A^a_\nu) + \varepsilon \epsilon^{abc}[\partial^\nu (A^b_\nu A^c_\mu) + A^b_\nu \partial^\nu A^c_\mu - A^b_\nu \partial_\mu A^{c\nu}] + \epsilon^{abc}\epsilon^{cde} A^{b\nu} A^d_\nu A^e_\mu = \varepsilon j^a_\mu \tag{3}$$

where ϵ^{abc} is the totally antisymmetric symbol and the metric tensor signature is (+++−). It should also be remembered throughout the paper that potentials and sources are time independent and that all time derivatives vanish. Thus the equations (3) are the elliptic semilinear partial differential equations†.

Substituting (1) into (3) in nth order in ε simpler equations are obtained:

$$M^{ab\nu}_\mu \overset{n}{A}{}^b_\nu = \overset{n-1}{J}{}^a_\mu \tag{4}$$

where the 12×12 matrix M contains the zero-order coefficients only:

$$M^{ab\nu}_\mu \equiv \epsilon^{abc}\epsilon^{cde} \overset{0}{A}{}^e_\mu \overset{0}{A}{}^{d\nu} + \epsilon^{adc}\epsilon^{ceb} \overset{0}{A}{}^d_\lambda \overset{0}{A}{}^{e\lambda} \delta^\nu_\mu + \epsilon^{adc}\epsilon^{cbe} \overset{0}{A}{}^e_\mu \overset{0}{A}{}^{d\nu} \tag{5}$$

and

$$\begin{aligned} \overset{n-1}{J}{}^a_\nu \equiv{}& \overset{n-1}{j}{}^a_\nu - \Delta \overset{n-2}{A}{}^a_\nu + \partial_\nu \partial^\mu \overset{n-2}{A}{}^a_\mu - \sum_{i=0}^{n-1} [\partial^\mu (\epsilon^{abc} \overset{i}{A}{}^b_\mu \overset{n-i-1}{A}{}^c_\nu) \\ & + \epsilon^{abc} \overset{i}{A}{}^{b\mu} (\partial_\mu \overset{n-i-1}{A}{}^c_\nu - \partial_\nu \overset{n-i-1}{A}{}^c_\mu)] \\ & - \sum_{\substack{i,j=0 \\ i+j \leqslant n}}^{n-1} \epsilon^{abc}\epsilon^{cde} \overset{i}{A}{}^b_\mu \overset{j}{A}{}^{b\mu} \overset{n-i-j}{A}{}^e_\nu . \end{aligned} \tag{6}$$

Hence instead of the single equation (3) for the full potential we have a chain of recurrence equations (4) for coefficients $\overset{0}{A}, \overset{1}{A}, \overset{2}{A}, \ldots$.

In zero and first order (4) yields simple equations‡:

$$\overset{0}{\mathbf{A}}{}^\mu \times (\overset{0}{\mathbf{A}}_\mu \times \overset{0}{\mathbf{A}}_\nu) = 0 \tag{7}$$

$$M^{ab\nu}_\mu \overset{1}{A}{}^b_\nu = \overset{0}{j}{}^a_\mu - \partial^\nu (\epsilon^{abc} \overset{0}{A}{}^b_\nu \overset{0}{A}{}^c_\mu) + \epsilon^{abc} \overset{0}{A}{}^{b\nu} (\partial_\nu \overset{0}{A}{}^c_\mu - \partial_\mu \overset{0}{A}{}^c_\nu). \tag{8}$$

The solution of equation (7) is

$$\overset{0}{A}{}^a_\mu = A^a(x) \alpha_\mu(x) \tag{9}$$

where $A^a(x)$, $\alpha_\mu(x)$ are arbitrary functions. Notice that (9) is of generalised Maxwell-type (GMT) form (Górski 1982).

Using (9) it can be checked by direct calculation that the 12×12 matrix M is singular and that

$$\text{rang} \|M\| = 6. \tag{10}$$

† Notice that a small parameter multiplies the differential operator Δ. This is similar to the JWKB approximation for the Schrödinger equation with the Planck constant $\hbar$ instead of ε (Schiff 1968). But in quantum mechanics the $\hbar$ expansion for a wavefunction is highly singular and the zero-order approximation cannot be obtained by the simple substitution $\hbar = 0$. In this context non-linear equations seem to be more plausible than linear ones (Berger 1977).

‡ Vectors are in group ('colour') space.

This is a plausible feature because it prevents equations (4) from being overdetermined. It should be stressed that if the series (1) is truncated at the Nth term ($n = N$) then (4) implies $12(3N+1)$ constraints on the $12(N+1)$ coefficients $\overset{n}{A}{}^a_\mu$†. This is because, in general, $\overset{N}{J}, \ldots, \overset{3N-1}{J}$ do not vanish even though $\overset{N+1}{A} = \overset{N+2}{A} = \ldots = 0$ (see the definition (6)). Hence (4) implies $12 \times 2N$ additional equations: $\overset{N+1}{J} = 0$, $\overset{N+2}{J} = 0, \ldots, \overset{3N-1}{J} = 0$ (for $n \geqslant 3N, \overset{n}{J}$ vanish identically, if truncated). According to (10) some of the first $12(N+1)$ equations are linear combinations of the others. Thus the chain of recurrence equations becomes less overdetermined.

Remaining superfluous equations may be viewed as additional constraints on the external current and due to them higher-order coefficients in the expansion (1) vanish in the sourceless case. These constraints can be easily seen in explicit examples of solutions which will be given in a separate paper.

With the aid of (9) the first-order equation (8) can be reduced to a simpler one:

$$M^{ab\nu}_\mu \overset{1}{A}{}^b_\nu = \overset{0}{j}{}^a_\mu + \epsilon^{abc} \overset{0}{A}{}^{b\nu} (\partial_\nu \overset{0}{A}{}^c_\mu - \partial_\mu \overset{0}{A}{}^c_\nu). \tag{8'}$$

If expansion (1) is truncated at the zero-order term $\overset{0}{A}{}^a_\mu(x)$ then (8′) can be solved after imposing appropriate boundary conditions.

Furthermore, in the limit $\varepsilon \to 0$ it is obvious that $\overset{0}{A}{}^a_\mu(x)$ transforms homogeneously under the gauge transformation $\omega(x)$;

$$\overset{0}{A}{}'_\mu = \omega^{-1} \overset{0}{A}_\mu \omega + O(\varepsilon), \tag{11}$$

where matrix notation is used. Hence the form (9) of the strong-coupling limit of the YM potentials is gauge invariant and the zero-order equation (7) is gauge covariant. Gauge covariance in arbitrary order will be proved in § 5.

3. Mathematical background

The mathematical problem we are dealing with is called a *singular perturbation problem* (Berger 1977).

When a small parameter multiplies a differential operator, the asymptotic ε expansion of the solution is, in general, non-analytic and includes negative powers of ε which have to be added to fulfil boundary conditions. Unfortunately there is no mathematical theory of singular perturbations applicable to *non-linear partial* differential equations. Only special results have been obtained for simple equations of this kind.

First some basic mathematical concepts will be reviewed.

Definition 1. Let $F_\varepsilon[x]$ be a C^1 mapping of Banach spaces X into Y depending continuously on a small parameter ε and let there be a sequence $x_n(\varepsilon) \in X$ ($n = 0, 1, 2, \ldots, N$) such that

(i) $\|F_\varepsilon[x_n(\varepsilon)]\| = O(\varepsilon^{n+1})$ for fixed n, as $\varepsilon \to 0$ and

(ii) for small non-zero ε there is a solution $\bar{x}(\varepsilon)$ of $F_\varepsilon[x] = 0$ such that $\|\bar{x}(\varepsilon) - x_n(\varepsilon)\| = O(\varepsilon^{n+1})$ for n fixed, $\varepsilon \to 0$.

Under these circumstances we say that $x_n(\varepsilon)$ is an *asymptotic approximation* to the solution $\bar{x}(\varepsilon)$.

† In the special case $N = 0$, 3×12 equations are obtained.

Clearly in many applications the sequence $\{\|x_n(\varepsilon)\|\}$ diverges as $n \to \infty$ for fixed $\varepsilon \neq 0$ (this is why the approximation is called 'asymptotic').

For our purposes a more restrictive approximation will be useful.

Definition 2. The asymptotic approximation of the form

$$x_n(\varepsilon) = \sum_{i=0}^{n} \alpha_i \varepsilon^i \tag{12}$$

where α_i may depend on ε but $\|\alpha_i\|$ is bounded independently of ε is called an *asymptotic expansion.*

We are going to interpret (1) as an asymptotic expansion of the true solution to the YM equations. It satisfies condition (i) (see the lemma below) but we cannot prove the existence condition (ii) in general. Nevertheless solutions which have the exact (i.e. not only asymptotic) form (1) certainly obey this condition.

Rigorous results concerning the applicability of the asymptotic expansion were obtained for the *semilinear Dirichlet problem* which is, in a sense, similar to our problem and is defined below:

$$\varepsilon^2 \Delta\varphi + f(x, \varphi, \varepsilon) = 0 \tag{13}$$

$$\varphi(x, \varepsilon) = h(x) \qquad \text{on } \partial\Omega \tag{13'}$$

where the bounded domain $\Omega \subset \mathbb{R}^N$. In the special case when†

$$f(x, \varphi, \varepsilon) = \varphi(x) - g^2(x)\varphi^3(x) \tag{14}$$

and

$$h(x) = 0 \tag{14'}$$

we have the following theorem.

Theorem 1. For sufficiently small values of ε the problem (13), (14) has a unique smooth positive solution $\varphi(x, \varepsilon)$ which tends to $1/g(x)$ as $\varepsilon \to 0$ outside a narrow boundary layer of width $O(\varepsilon)$ concentrated near $\partial\Omega$. To each order in ε appropriate boundary layer corrections can be found for φ to satisfy the boundary value problem. The proof can be found in the literature (Berger and Fraenkel 1970).

The following theorem was proved for more general functions $f(x, \varphi, \varepsilon)$ (Fife 1973).

Theorem 1'. Theorem 1 can be generalised for sufficiently smooth functions $f(x, \varphi, \varepsilon)$ and $h(x)$ if there exists a function $\overset{0}{\varphi}(x)$ satisfying (i) $f(x, \overset{0}{\varphi}(x), 0) = 0$, (ii) $f_\varphi(x, \overset{0}{\varphi}(x), 0) > 0$ and at each point $x \in \partial\Omega$ we have (iii) $\int_{\overset{0}{\varphi}(x)}^{k} f(x, \varphi, 0)\, d\varphi > 0$ for all $k \neq \overset{0}{\varphi}(x)$ in the closed interval bounded by $\overset{0}{\varphi}(x)$ and $h(x)$.

The requirements (i) and (ii) are imposed to assure the existence of the zero-order approximation $\overset{0}{\varphi}(x)$ and to determine recursively and uniquely $\overset{m}{\varphi}(x)$ by means of $\overset{i}{\varphi}(x)$ $(i < m)$. Condition (iii) allows us to construct the appropriate boundary layer correction.

† $g(x)$ is a smooth, strictly positive function of Ω.

Thus asymptotic expansions can be used successfully for a wide class of elliptic semilinear equations of the type (13). Notice that (13) with the particular form of $f(x, \varphi, \varepsilon)$ given by (14) is a field equation for the φ^4 theory with $g^2 \equiv \lambda/m^2$ and x rescaled: $x \to \varepsilon m x$.

Unfortunately we have no rigorous proof of to what extent the above theorems can be applied to the YM (static or Euclidean) equations which are also elliptic and semilinear. This is a very difficult mathematical problem.

The asymptotic expansion can be used for the YM equations even without any boundary layer correction, at least for special sources and solutions. This can be seen from the fact that, for any potential which is analytic at $\varepsilon = 0$ and suitably regular in x, the *source* can be constructed explicitly so as to satisfy equations (4).

We now prove a simple lemma concerning the formal expansion (1). Let an operator O_{YM} be the operator acting on potentials A^a_μ in such a way that $O_{\mathrm{YM}}[A^a_\mu] = 0$ is precisely the YM equation (3). In addition, let $\tilde{A}^a_\mu$ be defined by

$$\tilde{A}^a_\mu \equiv \sum_{n=0}^{M} \overset{n}{A}{}^a_\mu \varepsilon^n. \tag{15}$$

Lemma. $\tilde{A}^a_\mu$ by construction satisfies the YM equations to the Mth order in ε, i.e. there exists a positive constant K depending only on $\overset{0}{A}{}^a_\mu$ such that

$$\|O_{\mathrm{YM}}[\tilde{A}^a_\mu](x, \varepsilon)\|_{\sup} \leqslant K\varepsilon^{M+1} \qquad \text{for all } x \tag{16}$$

where $\|\cdot\|_{\sup}$ is a supremum norm.

Proof. $\overset{n}{A}{}^a_\mu(x)$ were determined by the requirement

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} O_{\mathrm{YM}}[\tilde{A}^a_\mu]|_{\varepsilon=0} = 0 \tag{17}$$

where $0 \leqslant n \leqslant M$. Equation (17) is equivalent to the recurrence equation (4). Hence, from Taylor's theorem, we have immediately

$$O_{\mathrm{YM}}[\tilde{A}^a_\mu](x, \varepsilon) = \varepsilon^{M+1} \frac{1}{(M+1)!} O_{\mathrm{YM}}[\tilde{A}^a_\mu](x, \theta\varepsilon) \qquad 0 < \theta < 1. \tag{18}$$

Because all derivatives are bounded the lemma is proved.

Notice that the regularity assumptions are essential in the above proof. For this reason our approach is especially suitable for smooth, soliton-like potentials rather than for singular ones.

It should be stressed that the lemma proved is connected with the formal expansion (1) and does not solve the existence problem. Instead of a rigorous proof two simple examples will be given in the following section to support the application of the method presented to the YM equations.

4. Examples

Equation (13) is similar to but not the same as the YM equation, which is more complicated. The main qualitative differences are that the YM equation is in fact a set of equations and in the zero-order approximation they are reduced to a set of algebraic *indeterminate* equations. Moreover the YM equations contain the *first-order derivatives* multiplied by ε†. Simple examples will be given to show that the formal ε expansion can be used successfully for solutions to equations with such properties.

Example 1. We shall start with a set of two non-linear (algebraic) equations of the form

$$\varepsilon x^2+x-y-1=0 \tag{19a}$$

$$\varepsilon y^2+x-y-1=0. \tag{19b}$$

Because of their simplicity, equations (19) can be solved exactly. They have the following four solutions:

$$x_{\rm I}=\varepsilon^{-1}[-1+(1+\varepsilon)^{1/2}] \qquad y_{\rm I}=\varepsilon^{-1}[+1-(1+\varepsilon)^{1/2}] \tag{20a}$$

$$x_{\rm II}=\varepsilon^{-1}[-1-(1+\varepsilon)^{1/2}] \qquad y_{\rm II}=\varepsilon^{-1}[+1+(1+\varepsilon)^{1/2}] \tag{20b}$$

$$x_{\rm III}=+\varepsilon^{-1/2} \qquad y_{\rm III}=+\varepsilon^{-1/2} \tag{20c}$$

$$x_{\rm IV}=-\varepsilon^{-1/2} \qquad y_{\rm IV}=-\varepsilon^{-1/2}. \tag{20d}$$

Of these solutions, only (20*a*) has a finite limit as $\varepsilon\to 0$. In this limit it tends to $\pm\frac{1}{2}$ respectively:

$$x_{\rm I}(\varepsilon)\to\overset{0}{x}=+\tfrac{1}{2}, \qquad y_{\rm I}(\varepsilon)\to\overset{0}{y}=-\tfrac{1}{2}. \tag{21}$$

This solution can be obtained with the aid of an ε expansion in a form analogous to (1):

$$x=\sum_{n=0}^{\infty}\overset{n}{x}\varepsilon^n \qquad y=\sum_{n=0}^{\infty}\overset{n}{y}\varepsilon^n. \tag{22}$$

In the zero-order approximation we get only one equation for two variables:

$$\overset{0}{x}-\overset{0}{y}-1=0. \tag{23}$$

This equation has an infinite family of solutions for the zero-order coefficients in (22). Notice that (21) is one of them. Moreover, this solution can be determined exactly when the first-order equations analogous to (4) are used. They have the form

$$\overset{0}{x}{}^2+\overset{1}{x}-\overset{1}{y}=0 \tag{24a}$$

$$\overset{0}{y}{}^2+\overset{1}{x}-\overset{1}{y}=0. \tag{24b}$$

The consistency condition for equations (24) is

$$\overset{0}{x}{}^2=\overset{0}{y}{}^2 \tag{24'}$$

and picks out exactly the true solution (21) from the infinite family of all solutions to (23). In the same way highest-order coefficients can be computed.

Example 2. In this example a non-linear and non-homogeneous equation with the first-order derivatives multiplied by a small parameter ε will be considered. It has the form

$$\varepsilon^2 ff''+2\varepsilon(f-1)f'=\varepsilon^4/(x-\varepsilon)^4 \tag{25}$$

† Due to this feature the small parameter ε can be rescaled from the left-hand side of the YM equations by transformation of the variable $x\to\varepsilon x$. The equation considered in the second example will have the same property.

where $f(x)$ has an infinite domain $\Omega=\{\mathbb{R}\ni x : x \geqslant \text{constant} > \varepsilon\}$. We are looking for a solution to (25) in the form

$$f(x,\varepsilon)=\overset{0}{f}(x)+\varepsilon\overset{1}{f}(x)+\varepsilon^2\overset{2}{f}(x)+\ldots. \tag{26}$$

In the two lowest orders the recurrence equation implies

$$2(\overset{0}{f}-1)\overset{0}{f}{}'=0 \tag{27a}$$

$$\overset{0}{f}\overset{0}{f}{}''+2(\overset{0}{f}-1)\overset{1}{f}{}'+2\overset{1}{f}\overset{0}{f}{}'=0. \tag{27b}$$

The first of these has the solution

$$\overset{0}{f}(x)=1 \tag{28a}$$

and then (27*b*) is satisfied automatically.

The next two equations are:

$$\overset{1}{f}{}''+2\overset{1}{f}\overset{1}{f}{}'=0 \tag{27c}$$

$$\overset{2}{f}{}''+\overset{1}{f}{}''\overset{1}{f}+2\overset{1}{f}\overset{2}{f}{}'+2\overset{2}{f}\overset{1}{f}{}'=2/x^4 \tag{27d}$$

where (28*a*) has been taken into account. The solution to (27*c*), which is regular in Ω, is

$$\overset{1}{f}(x)=1/x. \tag{28b}$$

In the same way, using (27*d*) and (28*b*), we have

$$\overset{2}{f}(x)=1/x^2. \tag{28c}$$

Hence an approximate solution to (25) is

$$f(x,\varepsilon)=1+\frac{\varepsilon}{x}+\frac{\varepsilon^2}{x^2}+\ldots. \tag{29}$$

From (29) it can be guessed that the expansion (29) has the compact form

$$f(x,\varepsilon)=(1-\varepsilon/x)^{-1}. \tag{30}$$

It is easy to check that (30) is actually an exact solution to (25) and our method is adequate.

We have seen in the above examples that, at least for some of the solutions to the non-linear equations with properties similar to these of the YM equations, the ε expansion is correct and directly applicable (i.e. even boundary layer corrections and any 'regularisation' are unnecessary). The same conclusion may be drawn if we consider known solutions to the YM equations which have the form (1). Other solutions with a more complicated ε dependence, but analytic at $\varepsilon=0$, will be given explicitly elsewhere.

5. Gauge covariance

We now prove that the proposed approximation scheme is gauge invariant and that the recurrence equations (4) are gauge covariant. This is a non-trivial feature because a gauge transformation is ε dependent and its infinitesimal form is (Yang and Mills 1954)

$$\boldsymbol{A}'_\mu = \boldsymbol{A}_\mu + \boldsymbol{\theta} \times \boldsymbol{A}_\mu - \varepsilon\, \partial_\mu \boldsymbol{\theta} \tag{31}$$

where $\boldsymbol{\theta} = \boldsymbol{\theta}(x)$ are infinitesimal gauge functions.

For the sake of simplicity we introduce the brief notation

$$\{A\} \equiv \Delta \boldsymbol{A}_\mu - \partial_\mu \partial^\nu \boldsymbol{A}_\nu \tag{32a}$$

$$\{A, B\} \equiv \partial^\nu \boldsymbol{A}_\nu \times \boldsymbol{B}_\mu - \boldsymbol{A}_\nu \times \partial_\mu \boldsymbol{B}^\nu + 2\boldsymbol{A}_\nu \times \partial^\nu \boldsymbol{B}_\mu \tag{32b}$$

$$\{A, B, C\} \equiv \boldsymbol{A}^\nu \times (\boldsymbol{B}_\nu \times \boldsymbol{C}_\mu) \tag{32c}$$

where the brace symbols actually have one Minkowski and one gauge group ('colour') index which are suppressed in the definition (32).

In the notation (32) the YM equations are

$$\varepsilon^2\{A\} + \varepsilon\{A, A\} + \{A, A, A\} = \varepsilon j \tag{33}$$

and instead of (4), (5) and (6) we have

$$\{\overset{n-2}{A}\} + \sum_{i=0}^{n-1} \{\overset{n-i-1}{A}, \overset{i}{A}\} + \sum_{\substack{i,j=0\\ i+j\le n}}^{n} \{\overset{i}{A}, \overset{j}{A}, \overset{n-i-j}{A}\} = \overset{n-1}{j}. \tag{34}$$

The full YM equations (33), due to their gauge covariance, have the following form in our notation if an infinitesimal gauge transformation is applied:

$$\varepsilon^2\{A\} + \varepsilon\{A, A\} + \{A, A, A\} + \boldsymbol{\theta} \times [\varepsilon^2\{A\} + \varepsilon\{A, A\} + \{A, A, A\}] = \varepsilon j + \boldsymbol{\theta} \times \varepsilon j \tag{33'}$$

where terms bilinear in $\boldsymbol{\theta}$ are neglected.

Hence, after a gauge transformation, instead of (34) the following equation is obtained:

$$\begin{aligned}
&\{\boldsymbol{\theta} \times \overset{n-2}{A}\} - \delta_{n3}\{\partial\boldsymbol{\theta}\} + \sum_{i=0}^{n-1} \{\overset{n-i-1}{A}, \boldsymbol{\theta} \times \overset{i}{A}\} + \sum_{i=0}^{n-1} \{\boldsymbol{\theta} \times \overset{n-i-1}{A}, \overset{i}{A}\} - \{\overset{n-2}{A}, \partial\boldsymbol{\theta}\} - \{\partial\boldsymbol{\theta}, \overset{n-2}{A}\} \\
&\quad + \sum_{\substack{i,j=0\\ i+j\le n}}^{n} \{\overset{i}{A}, \overset{j}{A}, \boldsymbol{\theta} \times \overset{n-i-j}{A}\} + \sum_{\substack{i,j=0\\ i+j\le n}}^{n} \{\overset{i}{A}, \boldsymbol{\theta} \times \overset{j}{A}, \overset{n-i-j}{A}\} + \sum_{\substack{i,j=0\\ i+j\le n}}^{n} \{\boldsymbol{\theta} \times \overset{i}{A}, \overset{j}{A}, \overset{n-i-j}{A}\} \\
&\quad - \sum_{i=0}^{n-1} \{\overset{i}{A}, \overset{n-i-j}{A}, \partial\boldsymbol{\theta}\} - \sum_{i=0}^{n-1} \{\overset{i}{A}, \partial\boldsymbol{\theta}, \overset{n-i-1}{A}\} - \sum_{i=0}^{n-1} \{\partial\boldsymbol{\theta}, \overset{i}{A}, \overset{n-i-1}{A}\} \\
&= \boldsymbol{\theta} \times \overset{n-1}{j};
\end{aligned} \tag{34'}$$

only terms linear in $\boldsymbol{\theta}$ are taken into account.

Using the algebraic formulae for the brace symbols given in the appendix it can be easily proved that equation (34′) is equivalent to

$$\boldsymbol{\theta} \times \left(\{\overset{n-2}{A}\} + \sum_{i=0}^{n-1} \{\overset{n-i-1}{A}, \overset{i}{A}\} + \sum_{\substack{i,j=0\\ i+j\le n}}^{n} \{\overset{i}{A}, \overset{j}{A}, \overset{n-i-j}{A}\} \right) = \boldsymbol{\theta} \times \overset{n-1}{j}. \tag{35}$$

This means that after the gauge transformation (31) of a potential the recurrence equation (34) is unchanged (it transforms homogeneously in the matrix notation). Hence we have the following theorem.

Theorem 2. Equations (34) are gauge covariant in each order and transform homogeneously under a gauge transformation, in the same way as the YM equations (33) do.

6. Screening effect

It was suggested (Mandula 1977) that, in analogy to scalar massless electrodynamics, in the YM theory an external source can be screened for sufficiently large values of the coupling constant. Here it will be shown that the screening effect occurs in the YM theory in the limit $g \to \infty$.

The YM field is charged and its current is (Yang and Mills 1954)

$$j_\mu^{\text{field}} \equiv g\boldsymbol{A}^\nu \times \boldsymbol{F}_{\mu\nu} \tag{36}$$

and the total current is

$$\boldsymbol{J}_\mu \equiv j_\mu^{\text{field}} + \boldsymbol{j}_\mu. \tag{37}$$

The currents (36) and (37) are obviously gauge dependent but the total current (37) is conserved:

$$\partial^\mu \boldsymbol{J}_\mu = 0. \tag{38}$$

Cancellation of the external current by the field current is called the local screening effect. In the case of total screening we have

$$\boldsymbol{J}_\mu = 0. \tag{39}$$

There is also a possible weaker effect, the so called global screening effect. This takes place only when external and field charges cancel each other. However, in general, both effects are gauge dependent and may have no physical relevance (Hughes 1979).

The total screening condition (39) can be expressed in terms of $\overset{i}{A}{}_\mu^a$ coefficients and brace symbols. Using the ε expansion we have an nth-order equation of the form

$$\sum_{i=0}^{n-1} \overset{n-i-1}{\boldsymbol{A}}{}^\mu \times (\partial_\mu \overset{i}{\boldsymbol{A}}_\nu - \partial_\nu \overset{i}{\boldsymbol{A}}_\mu) + \sum_{\substack{i,j=0 \\ i+j \leqslant n}}^{n} \overset{i}{\boldsymbol{A}}{}^\mu \times (\overset{j}{\boldsymbol{A}}_\mu \times \overset{n-i-j}{\boldsymbol{A}}_\nu) = \overset{n-1}{\boldsymbol{j}}_\nu . \tag{40}$$

Then the nth-order equation (34) implies additionally

$$\{\overset{n-2}{A}\} + \partial^\mu \sum_{i=0}^{n-1} \overset{i}{\boldsymbol{A}}_\mu \times \overset{n-i-1}{\boldsymbol{A}}_\nu = 0. \tag{41}$$

Thus in zero order equation (40) and the recurrence equation (34) are equivalent (according to (1) equation (41) is then an identity). As a result we have the following theorem.

Theorem 3. In zero order in $1/g$ the YM equations (34) with potentials of the form (1) imply the total screening effect. According to theorem 2 this result is *gauge independent.*

In higher orders in ε the rather restrictive condition (41) should be taken into account and the total screening may be broken down.

7. Conclusions and perspectives

In this paper a new approach has been proposed to deal with the classical YM theory in the large coupling constant region. An asymptotic expansion in $\varepsilon = 1/g$ was used to approximate solutions to the YM equations in the strong-coupling limit.

The validity of the asymptotic expansion for the semilinear elliptic equations has been reviewed briefly. Subsequently strict results were extended to the YM theory. This extension was also supported by simple examples. Moreover many explicitly known solutions have the expansion (1), the existence of which is crucial within the framework of our approach. Furthermore expansion (1) is very general in comparison with them.

The chain of recurrence equations (4) has some simple properties (e.g. they are totally algebraic in zero order in ε and algebraic with respect to the highest coefficient $\overset{i}{A}{}^a_\mu$ in each order) and its consistency conditions impose extra restrictions on an external current.

The recurrence equations were shown to be gauge covariant and to transform homogeneously under a gauge transformation.

It was also shown that in the strong-coupling limit the total local screening effect occurs at least in lowest order in the ε expansion. According to theorem 2 this effect is gauge invariant. Hence one can speculate that it is a physically relevant phenomenon. It seems to be rather encouraging after unpleasant conclusions reached previously (see, e.g., Hughes 1979).

At this stage investigations should be continued in two directions. Rigorous results connected with the range of mathematical correctness of the approximation schemes presented are important, especially the role of the form of the external current which regulates the ε expansion for potentials. Solvability and consistency conditions for the recurrence formula are source dependent and these problems were also not worked out.

On the other hand, the physical relevance should be analysed in detail, the recurrence equations should be solved, at least to the lower orders, and explicit non-trivial solutions with a finite strong-coupling limit should be constructed. This programme will be developed soon.

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Appendix

Here some useful properties of the brace symbols defined in § 5 are given.

For the linear symbol we have

$$\{\partial \boldsymbol{\theta}\} = 0 \tag{A1}$$

$$\{\boldsymbol{\theta} \times \overset{i}{A}\} - \{\overset{i}{A}, \partial\boldsymbol{\theta}\} - \{\partial\boldsymbol{\theta}, \overset{i}{A}\} = \boldsymbol{\theta} \times \{\overset{i}{A}\}. \tag{A2}$$

For the bilinear symbol

$$\sum_{\mathrm{P}} \{\overset{i}{A}, \boldsymbol{\theta}\times\overset{j}{A}\} + \sum_{\mathrm{P}} \{\boldsymbol{\theta}\times\overset{i}{A}, \overset{j}{A}\} = \boldsymbol{\theta}\times\sum_{\mathrm{P}} \{\overset{i}{A}, \overset{j}{A}\} + \sum_{\mathrm{P}} \{\overset{i}{A}, \overset{j}{A}, \partial\boldsymbol{\theta}\} \qquad \text{(A3)}$$

where $\sum_{\mathrm{P}}$ is a sum over permutations of the arguments in brace symbols.

Finally, for the trilinear brace symbol we have the following identity:

$$\sum_{\mathrm{P}} \{\overset{i}{A}, \overset{j}{A}, \boldsymbol{\theta}\times\overset{k}{A}\} + \sum_{\mathrm{P}} \{\overset{i}{A}, \boldsymbol{\theta}\times\overset{j}{A}, \overset{k}{A}\} + \sum_{\mathrm{P}} \{\boldsymbol{\theta}\times\overset{i}{A}, \overset{j}{A}, \overset{k}{A}\} = \boldsymbol{\theta}\times\sum_{\mathrm{P}} \{\overset{i}{A}, \overset{j}{A}, \overset{k}{A}\}. \qquad \text{(A4)}$$

The above identities can be proved by direct calculation. Using (A1)–(A4) equation (35) can be easily obtained from equation (34′).

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